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INFLUENCE OF DISSIPATION ON THE PROPAGATION OF A SPHERICAL EXPLOSION SHOCK WAVE*

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The problem of the propagation of an explosion shock wave in a weakly compressible viscous medium at low Reynolds numbers is solved by the method of asymptotic expansions. The influence of non-linear terms in the principal approximation is studied, and the law of wave amplitude damping and its profile are found.

1. Formulation of the problem. The system of equations that describes the spherically symmetric motion of a compressible viscous fluid is /1/

$$\frac{\partial\bar u}{\partial\bar t}+\bar u\frac{\partial\bar u}{\partial\bar x}=-\frac{1}{\bar\rho}\frac{\partial\bar p}{\partial\bar x}+\frac{1}{\bar\rho}\left(\zeta+\frac{4}{3}\mu\right)\left(\frac{\partial^2\bar u}{\partial\bar x^2}+\frac{2}{\bar x}\frac{\partial\bar u}{\partial\bar x}-\frac{2\bar u}{\bar x^2}\right) \tag{1.1}$$

$$\frac{\partial\bar\rho}{\partial\bar t}+\bar u\frac{\partial\bar\rho}{\partial\bar x}+\bar\rho\left(\frac{\partial\bar u}{\partial\bar x}+\frac{2\bar u}{\bar x}\right)=0$$

$$\frac{\partial\bar s}{\partial\bar t}+\bar u\frac{\partial\bar s}{\partial\bar x}=\frac{1}{\bar\rho\bar T}\left\{\left(\zeta-\frac{2}{3}\mu\right)\left(\frac{\partial\bar u}{\partial\bar x}+\frac{2\bar u}{\bar x}\right)^2+2\mu\left[\left(\frac{\partial\bar u}{\partial\bar x}\right)^2+\frac{2\bar u^2}{\bar x^2}\right]+\frac{\bar\varkappa}{\bar x^2}\frac{\partial}{\partial\bar x}\left(\bar x^2\frac{\partial\bar T}{\partial\bar x}\right)\right\}$$

where the bar refers to dimensional quantities, $\bar s, \bar T$ are the entropy per unit mass and the temperature, $\zeta, \mu, \bar\varkappa$ are the coefficients of shift, spatial viscosity, and thermal conductivity. Knowing the internal energy as a function of $\bar\rho$ and $\bar s$ we can find the dependences $\bar p(\bar\rho, \bar s)$ and $\bar T(\bar\rho, \bar s)$. These relations close system (1.1).

The action of the explosion products on a fluid is modelled by a piston, moving according to the law $\bar x=\bar\varphi(\bar t)$, where $\bar\varphi(0)=\bar x_0, \bar\varphi'(0)=\bar U_0$ ($\bar U_0$ is the shock initial velocity).

We will introduce the dimensionless variables

$$\sigma=\frac{\bar p}{\bar\rho_0\bar U_0^2},\quad \rho=\frac{\bar\rho}{\bar\rho_0},\quad u=\frac{\bar u}{\bar U_0},\quad x=\frac{\bar x}{\bar x_0},\quad t=\frac{\bar t\bar U_0}{\bar x_0},\quad T=\frac{\bar T}{\bar T_0},\quad s=\frac{\bar s\bar T_0}{\bar U_0^2}$$

where $\bar\rho_0$ and $\bar T_0$ are the density and temperature of the undisturbed medium.

The medium is assumed to be weakly compressible. We will introduce the small parameter $\varepsilon=(\partial\sigma/\partial\rho)_s^{-1}$ in the undisturbed medium, and solve the problem in the range of parameters ensuring small density disturbances: $\rho=1+\varepsilon p$. For inviscid flow, p is the same in the principal approximation as the dimensionless pressure /2/.

We shall seek the principal term of the expansion of the solution with respect to the small parameter ε. Neglecting terms in (1.1) that are obviously taken into account in later approximations, we obtain the system

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x}-\left(\frac{\partial\sigma}{\partial s}\right)_\rho\frac{\partial s}{\partial x}+\alpha^0\left(\frac{\partial^2u}{\partial x^2}+\frac{2}{x}\frac{\partial u}{\partial x}-\frac{2u}{x^2}\right) \tag{1.2}$$

$$\varepsilon\frac{\partial p}{\partial t}+\varepsilon u\frac{\partial p}{\partial x}+\frac{\partial u}{\partial x}+\frac{2u}{x}=0$$

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}=\frac{1}{T}\left\{\alpha_1^0\left(\frac{\partial u}{\partial x}+\frac{2u}{x}\right)^2+\alpha_2^0\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{2u^2}{x^2}\right]+\varkappa\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial T}{\partial x}\right)\right\}$$

$$\alpha_1^0=\frac{3\zeta-2\mu}{3\bar\rho_0\bar U_0\bar x_0},\quad \alpha_2^0=\frac{2\mu}{\bar\rho_0\bar U_0\bar x_0},\quad \alpha^0=\alpha_1^0+\alpha_2^0,\quad \varkappa=\frac{\bar\varkappa\bar T_0}{\bar\rho_0\bar U_0^3}$$

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The initial conditions are homogeneous, and the boundary condition is: $x = \varphi(t)$, $u = \varphi'(t)$.

Since we shall be considering small density disturbances, let us say something about the acoustic approach to this problem, developed in /3/, where the impulse evolution is studied in the light of non-linearity, non-equilibrium, and dissipation, including the joint action of viscosity and heat conduction. The initial profile of the shock wave (SW) is obtained experimentally as a known exponential function /4/. In addition, the assumption is made that one of the Riemann invariants vanishes: $\bar{p} - \bar{c}_0\bar{\rho}_0\bar{u} = 0$ ($\bar{c}_0$ is the velocity of sound in the undisturbed medium). The basis for this approach is the small influence of viscosity on lengths of the order of the wave width, i.e., the inviscid flow is refined for large Reynolds numbers; this is important at large distances.

When obtaining the fundamental equation used in non-linear acoustics, the following is assumed /5/: 1) the disturbances are small for all parameters, in particular, the entropy, which varies as a result of heat conduction (the non-linear terms in the last of Eqs.(1.2) are thrown out; 2) a special form of the equation of state is used; 3) we take $\bar{p} = \bar{\rho}_0\bar{c}_0\bar{u}$. With these assumptions, system (1.2) can be reduced to a single equation.

It was shown in /2/ that, for an inviscid medium, the third condition does not hold in certain zones, notably in the zone of short times, when the SW profile is being built up. It is also shown in /2/ that, when there is no viscosity and heat conduction, analytic solution of the above problem gives precisely an exponential profile, while the third condition holds in the SW zone. Non-linear effects are studied in /6, 7/. The third condition holds for inviscid flow everywhere, only in the case of plane symmetry both for the shockless motion of the piston /8/ and in the presence of SW /2/.

Thus the acoustic approach is justified for a narrow zone around the SW, where the gradient of the disturbances is much greater than in the remaining zone, and does not provide a solution of the problem of the formation of an SW profile, or in the zone around the piston. In the present paper none of the three assumptions is made, nor are they valid in certain domains.

Our last remark concerns the range in which the theory is applicable. We know that, when the detonation wave leaves the contact surface, the pressure in the fluid depends, not on the absolute size of the charge, but only on the rate of detonation, i.e., on the power of the explosive material (EM) /9/. Hence the range in which asymptotic and acoustic theory are applicable is determined by the power of the EM, and not by the absolute size of the charge. An analytic solution /2, 10/ shows that the profiles of spherical SW for an inviscid and thermally non-conducting medium are similar with different absolute sizes of the same EM. A condensed EM of the trotyl type give, on the interface with water, pressures of the order of tens of thousands of atmospheres, at which the assumption of weak compressibility are invalid.

Below, we consider the case when the viscosity is significant in the principal approximation. This happens when

$$\alpha_1{}^0 = \alpha_1\varepsilon^{-1/2},\quad \alpha_2{}^0 = \alpha_2\varepsilon^{-1/2},\quad \alpha^0 = \alpha\varepsilon^{-1/2},\quad \alpha_i, \alpha \sim 1$$

2. Solution in the short-time zone. This zone is determined by the scales of the variables: $x \sim 1$, $t = \tau\varepsilon^{1/2} \sim \varepsilon^{1/2}$, $p = p^0\varepsilon^{-1/2} \sim \varepsilon^{-1/2}$, $u \sim 1, s \sim 1, T \sim 1$.

In the principal approximation we obtain the system of equations

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} = -\frac{\partial p^0}{\partial x} + \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} - \frac{2u}{x^2}\right) \\
&\frac{\partial p^0}{\partial \tau} + \frac{\partial u}{\partial x} + \frac{2u}{x} = 0 \\
&\frac{\partial s}{\partial \tau} = \frac{1}{T}\left\{\alpha_1\left(\frac{\partial u}{\partial x} + \frac{2u}{x}\right)^2 + \alpha_2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{2u^2}{x^2}\right]\right\} \\
&T(\rho, s) = T(1 + \varepsilon^{1/2}p^0, s) = T(1, s) = T(s)
\end{aligned} \tag{2.1}$$

The initial conditions are zero, and the boundary condition is $\varphi(\tau\varepsilon^{1/2}) = 1, U(\tau\varepsilon^{1/2}) = 1$, i.e., $u(1, \tau) = 1$.

The first two equations will be solved independently of the third, and then the entropy is found.

We make the Laplace transformation

$$V(x, a) = \int_0^\infty u(\tau, x)e^{-\tau a}\,d\tau,\quad P(x, a) = \int_0^\infty p^0(\tau, x)e^{-\tau a}\,d\tau$$

We find V from (2.1) in the light of the initial conditions, the solution damping condition as $x \to +\infty$, $\mathrm{Real}\, a > 0$, and the boundary condition; then we obtain P from the second equation of the system.

Putting $\theta = \alpha a$ and making a Laplace inversion, we obtain

$$u=\frac{1}{2\pi ix}\int\limits_{\Gamma}\left(1+\frac{\alpha\sqrt{1+\theta}}{x\alpha}\right)f_1(\theta,x,\tau,\alpha)\,d\theta \tag{2.2}$$

$$p=p^0\varepsilon^{-1/2}=\frac{\varepsilon^{-1/2}}{2\pi ix}\int\limits_{\Gamma}\frac{f_1(\theta,x,\tau,\alpha)\,d\theta}{\sqrt{1+\theta}},\quad f_1=$$

$$\frac{1}{\theta+\alpha\sqrt{1+\theta}}\exp\left[\frac{\theta}{\alpha}\left(\tau-\frac{x-1}{\sqrt{1+\theta}}\right)\right]$$

where Γ is a vertical contour in the complex plane of $\theta=\theta_0^1+i\theta_0^2$, located to the right of all singular points of the integrand. We indicate the branch of the root in (2.2): the solution is damped as $x\to+\infty$ if the branch of the root is defined by the cut along the semi-axis $\theta_0^2=0,\ -\infty<\theta_0^1<-1$, and the condition $\arg(1+\theta_0^1+i\infty)=\pi/2$.

The formally constructed solution (2.2) satisfies the equations, and the boundary and initial conditions.

It can be shown that, if the expressions for p^0 and u can be differentiated under the integral sign the required number of times, they will satisfy system (2.1). The uniform convergence of all the integrals encountered when checking the conditions with $t\geqslant 0,\ x\geqslant 1+\Delta\ (\Delta>0)$, which ensures in particular differentiation under the integral sign, follows from the bound

$$\left|\exp\left[-\frac{\theta}{\alpha}\left(\tau-\frac{x-1}{\sqrt{1+\theta}}\right)\right]\right|\leqslant A_1\exp\left(\frac{1-x}{\alpha\sqrt{\theta_0^2}}\right)$$

where A_1 is independent of x and τ.

We will check that the boundary condition holds for the velocity. By the uniform convergence of the integral,

$$u(1,\tau)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{1}{\theta}\exp\left[\frac{\theta}{\alpha}\left(\tau-\frac{1}{\sqrt{1+\theta}}\right)\right]d\theta=1$$

We will check that the initial conditions hold. By the uniform convergence, we can pass to the limit under the integral sign as $\tau\to 0$:

$$u(x,0)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{1}{\theta}\exp\left[-\frac{\theta(x-1)}{\alpha\sqrt{1+\theta}}\right]d\theta=\frac{1}{2\pi i}\lim_{M\to 0}\int\limits_{\Gamma_1}\frac{1}{\theta}\exp\left[-\frac{\theta(x-1)}{\alpha\sqrt{1+\theta}}\right]d\theta$$

(the contour Γ_1: $\{\theta=Me^{i\psi},\ -\pi/2\leqslant\psi\leqslant\pi/2\}$ is shown in Fig.1).

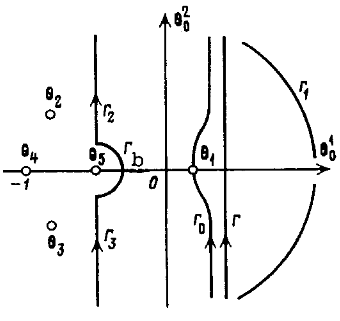

Fig.1

For the integral along Γ_1 we obtain the bound

$$\left|\int\limits_{\Gamma_1}\frac{1}{\theta}\exp\left[-\frac{\theta(x-1)}{\alpha\sqrt{1+\theta}}\right]d\theta\right|<2\int\limits_{-\pi/2}^{\pi/2}\exp\left[-\frac{\sqrt{M}(x-1)}{\alpha}\cos\frac{\psi}{2}\right]d\psi\leqslant$$

$$2\pi\exp\left[-\frac{\sqrt{M}(x-1)}{\alpha\sqrt{2}}\right]$$

since $\cos\psi/2\geqslant 1/\sqrt{2}$. For $x>1$, $M\to+\infty$, we obtain $u(x,0)=0$ (at $x=1,\ t=0$ a discontinuity). The initial condition is checked in the same way for the pressure.

Note that, in this statement, in the zone of short times, for the case of cylindrical symmetry and an inviscid medium, Laplace inversion leads to divergent integrals (even in the sense of the principal value).

2. Construction of the solution for $t\sim 1$. The scales of the variables in the zone round the piston are: $x\sim 1,\ u\sim 1$. In the principal approximation, the equation of continuity gives an incompressible fluid, for whose velocity we obtain $u=C(t)/x^2,\ C(t)=\varphi'(t)\varphi^2(t)$. Depending on the properties of the medium, i.e., on the functions $\sigma(\rho,s),\ T(\rho,s)$, the principal terms in the equations of motion and energy can be varied. Let the conditions $\partial\sigma/\partial s\sim 1,\ \partial T/\partial s\sim 1$ hold. Measuring the entropy from the entropy of the undisturbed state, we write the energy equation as

$$\frac{\partial G}{\partial t}+u\frac{\partial G}{\partial x}=\frac{6\alpha_2}{x^5}\varepsilon^{-1/2}C^2(t)+\frac{\varkappa}{x^2}\frac{\partial}{\partial x}\left[x^2\frac{\partial T(G)}{\partial x}\right],$$

$$G(s)=\int\limits_0^s T(s)\,ds$$

It is clear from this that $G=G^i\varepsilon^{-1/2},\ G^i\sim 1$. We obtain in the principal approximation

$$\frac{\partial G^i}{\partial t} + u\frac{\partial G^i}{\partial x} = 6\alpha_2\frac{C^2}{x^5} + \frac{\varkappa}{x^2}\frac{\partial}{\partial x}\left[x^2\frac{\partial T}{\partial G}\frac{\partial G^i}{\partial x}\right] \tag{3.1}$$

In the equation of motion, the viscous term vanishes. Noting that

$$\frac{\partial\sigma}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial\sigma}{\partial G}\frac{\partial G}{\partial x},$$

we obtain $p = p^i\varepsilon^{-1/3} \sim \varepsilon^{-2/3}$. As a result, we have

$$\frac{\partial p^i}{\partial x} + \left(\frac{\partial\sigma}{\partial s}\right)_\rho\left(\frac{\partial s}{\partial G}\right)_\rho\frac{\partial G^i}{\partial x} = 0 \tag{3.2}$$

The pressure rises as a result of the entropy variation. Integrating Eq.(3.2), we obtain

$$p = p^i\varepsilon^{-1/3} = -\int\frac{\partial\sigma(G)}{\partial G}dG = -\sigma(1, G) + g(t) \tag{3.3}$$

The function $g(t)$ will be found by union with the expansion describing the solution in the next zone.

When $\varkappa = 0$, the solution of Eq.(3.1) can easily be found by the method of characteristics in the parametric form

$$x = [\varphi^3(t) + n]^{1/3}, \quad G^i = 6\alpha_2\int\limits_0^t\frac{C^2(t_1)\,dt_1}{[\varphi^3(t_1) + n]^2}, \quad 0 \leqslant n < \infty \tag{3.4}$$

Let us construct the solution in the so-called wave zone, defined by the scales of the variables: $t \sim 1$, $x = x_e\varepsilon^{-1/3} \sim \varepsilon^{-1/3}$, $p = p_e\varepsilon^{-1/3} \sim \varepsilon^{2/3}$, $u = u_e\varepsilon \sim \varepsilon$, $s = s_e\varepsilon^{5/3} \sim \varepsilon^{5/3}$, $T \sim 1$. In the principal approximation, system (1.2) takes the form

$$\frac{\partial u_e}{\partial t} = -\frac{\partial p_e}{\partial x_e}, \quad \frac{\partial p_e}{\partial t} + \frac{\partial u_e}{\partial x_e} + \frac{2u_e}{x_e} = 0 \tag{3.5}$$

$$\frac{\partial s}{\partial t} = \frac{1}{T}\left\{\alpha_1\left(\frac{\partial u_e}{\partial x_e} + \frac{2u_e}{x_e}\right)^2 + \alpha_2\left[\left(\frac{\partial u_e}{\partial x_e}\right)^2\frac{2u_e^2}{x_e^2}\right]\right\}$$

The system splits up, and the entropy increases weakly due to viscosity. The general solution of system (3.5) is

$$p_e = \frac{F_1'(-x_e + t)}{x_e} + \frac{F_2'(x_e + t)}{x_e} \tag{3.6}$$

$$u_e = \frac{F_1(-x_e + t)}{x_e^2} + \frac{F_2(x_e + t)}{x_e^2} + \frac{F_1'(-x_e + t)}{x_e} - \frac{F_2'(x_e + t)}{x_e}$$

In the zone about the line $x - 1 = t\varepsilon^{-1/3}$, this solution cannot be combined with the solution for short times, so that another expansion has to be sought there. The arbitrary functions in (3.6) can be found only from the union, so that the boundary conditions for system (3.5) have to be replaced.

Let us construct the solution in the zone given by the scales: $\xi = \varepsilon^{1/4}(x - t\varepsilon^{-1/3} - 1) \sim 1$, $t \sim 1$, $p = \delta p_v \sim \delta$, $u = \varepsilon^{1/3}\delta u_v \sim \varepsilon^{1/3}\delta$, $x\varepsilon^{1/3}/t \approx 1$, $x_e = x\varepsilon^{1/3} \sim 1$ (ord $\delta \leqslant$ ord 1). The system in the principal approximation in the new coordinates is (the second-order terms have to be retained in order to find the principal approximation correctly)

$$-\frac{\partial u_v}{\partial\xi}\varepsilon^{1/4} + \frac{\partial u_v}{\partial t}\varepsilon^{1/3} = -\frac{\partial p_v}{\partial\xi}\varepsilon^{1/4} + \alpha\varepsilon^{1/6}\frac{\partial^2 u_v}{\partial\xi^2} - \left(\frac{\partial\sigma}{\partial s}\right)_\rho\frac{\partial s}{\partial\xi}\varepsilon^{1/4} \tag{3.7}$$

$$-\frac{\partial p_v}{\partial\xi}\varepsilon^{1/4} + \frac{\partial p_v}{\partial t}\varepsilon^{1/3} + \frac{\partial u_v}{\partial\xi}\varepsilon^{1/4} + \frac{2u_v}{t}\varepsilon^{1/3} = 0$$

$$-\frac{\partial s}{\partial\xi}\varepsilon^{1/4} + \frac{\partial s}{\partial t}\varepsilon^{1/3} = \frac{\alpha\delta^2}{T}\varepsilon^2\left(\frac{\partial u_v}{\partial\xi}\right)^2$$

It is clear from the last equation of (3.7) that $s \sim \varepsilon^{5/3}\delta^2$, i.e., the entropy term in the first equation of (3.7) has to be discarded in the principal approximation, while in the shock wave zone, the entropy increases only as a result of the viscosity.

The principal parts of the first and second equations of (3.7) are the same. Subtracting the first equation from the second, we obtain in the principal approximation

$$\frac{\partial u_v}{\partial\xi} = \frac{\partial p_v}{\partial\xi}, \quad \frac{\partial u_v}{\partial t} + \frac{\partial p_v}{\partial t} + \frac{2u_v}{t} = \alpha\frac{\partial^2 u_v}{\partial\xi^2} \tag{3.8}$$

The zone considered departs to infinity, where $p_v = u_v = 0$. Hence it follows from the first equation of (3.8) that the corresponding Riemann acoustic invariant vanishes: $u_v - p_v = 0$.

In the light of this, the second equation leads to the usual equation of heat conduction for the function $u_v t$. Its only solution which confines with the solution for short times is the fundamental solution

$$u_v t = p_v t = \frac{N}{\sqrt{t}} \exp\left(-\frac{\xi^2}{2t\alpha}\right) \quad (3.9)$$

The constants δ, N will be found from the union.

4. Intermediate asymptotic forms and union. The intermediate zone between $t \sim \varepsilon^{1/2}$ and $t \sim 1$ can be found using stretched time $t_* = t\varepsilon^{1/4} \sim 1$, i.e., is characterized by times $t \sim \varepsilon^{1/4}$.

Let us find the intermediate asymptotic form for the pressure in the initial zone. In the intermediate zone

$$p = p^0 \varepsilon^{-1/2} = \frac{\varepsilon^{-1/2}}{2\pi i x} \int_\Gamma \frac{\exp(\varepsilon^{-1/4}\alpha^{-1}t_* f(\theta, \lambda))}{\sqrt{1+\theta}\,(\theta + \alpha\sqrt{1+\theta})}\, d\theta \quad (4.1)$$

$$f(\theta, \lambda) = \theta\left(1 - \frac{\lambda}{\sqrt{1+\theta}}\right), \quad \lambda = \frac{x-1}{\tau}$$

The asymptotic behaviour of (4.1) can be found by the saddle-point method for fixed λ. The saddle points are found from the condition $\partial f/\partial\theta = 0$, which gives a cubic equation, and we find by Cardano's formula

$$\theta_1 = -1 + \frac{\lambda^2}{12} + D_1 + D_2, \quad \theta_{2,3} = -1 + \frac{\lambda^2}{12} - \frac{D_1 + D_2}{2} \pm i\frac{D_1 - D_2}{2}\sqrt{3}$$

$$D_1 = (-q/2 - Q)^{1/3}, \; D_2 = (-q/2 + Q)^{1/3}$$

$$q = -\frac{\lambda^2}{1728}(\lambda^4 + 36\lambda^2 + 216), \quad Q = \frac{\lambda^2}{24}\sqrt{9 + \lambda^2/3}$$

The first singular point of the integrand is $\theta_4 = -1$, while the second is given by the condition $-\theta/\alpha = \sqrt{1+\theta}$. Squaring both sides, we obtain a quadratic equation for θ. Noting the above choice of the branch of the root $\sqrt{1+\theta}$, we discard one root. As a result we find the second singular point $\theta_5 = (\alpha^2 - \alpha\sqrt{\alpha^2+4})/2 < 0$.

The mutual disposition of the saddle and singular points is shown in Fig.1. We have $\theta_1, \theta_5, \theta_4$ are real, and θ_2, θ_3 are complex conjugates; it can be shown that $-q/2 - Q \geqslant 0, D_1 \geqslant 0, D_2 \geqslant 0, D_1 \leqslant D_2, \mathrm{Real}\,\theta_1 \geqslant \mathrm{Real}\,\theta_{2,3}$; the point θ_4 is to the left of all the other points.

When finding the asymptotic forms in the problem, we can confine ourselves to passage through just the point θ_1, without passing through θ_2 and θ_3; the deformed contour Γ_0 is shown in Fig.1. The saddle direction at the point θ_1 is given by

$$k(\lambda) = \left(\frac{\partial^2 f}{\partial\theta^2}\right)_\lambda\Big|_{\theta_1} = \frac{(4+\theta_1)\lambda}{4(1+\theta_1)^{5/2}}$$

Obviously, $k(\lambda) \geqslant 0$, so that the saddle direction at θ_1 is vertical.

Depending on λ one of the conditions: $\theta_1 > \theta_5$, $\theta_1 < \theta_5$, $\theta_1 = \theta_5$, can hold. In the first case only the saddle point makes a contribution to the asymptotic form:

$$p_* = \frac{\varepsilon^{-3/8}}{x}\sqrt{\frac{\alpha}{2\pi k(\lambda)t_*}}\,\frac{\exp[\varepsilon^{-1/4}\alpha^{-1}t_* f(\theta_1(\lambda))]}{\sqrt{1+\theta_1}\,(\theta_1 + \alpha\sqrt{1+\theta_1})} \quad (4.2)$$

In the second case, on deformation, the contour covers the singular point θ_5, and we have to add to the right-hand side of (4.2) the residue at this point, equal to

$$\mathrm{res}(\theta_5) = \frac{\varepsilon^{-1/2}}{x}\,\frac{\exp[\varepsilon^{-1/4}\alpha^{-1}t_* f(\theta_5)]}{\sqrt{1+\theta_5}\,g(\theta_5)}, \quad g(\theta) = \frac{2\theta - \alpha^2 - \alpha\sqrt{\alpha^2+4}}{2(\theta - \alpha\sqrt{1+\theta})}$$

In the third case the saddle point merges completely with the pole of the integrand. This corresponds to a certain λ_0. To find the asymptotic form in this case, we have to deform the contour Γ_0, replacing it by the contour $\Gamma_3 + \Gamma_b + \Gamma_2$ (see Fig.1), where $\Gamma_2: \{\theta = \theta_5 + i\theta_0^2,\; b \leqslant \theta_0^2 < \infty\}$, $\Gamma_3: \{\theta = \theta_5 + i\theta_0^2,\; -\infty < \theta_0^2 \leqslant -b\}$, $\Gamma_b: \{\theta = \theta_5 + be^{i\psi},\; -\pi/2 \leqslant \psi \leqslant \pi/2\}$, $b \ll \varepsilon^{1/4}$.

By using the localization principle for integrals of the type considered /11/ and letting $b \to 0$, it can be shown that the asymptotic form of the integral (4.1) is

$$p_* = \frac{\varepsilon^{-1/2}}{2x\sqrt{1+\theta_5}\,g(\theta_5)} \exp[\varepsilon^{-1/4}\alpha^{-1}t_* f(\theta_5)] \quad (4.3)$$

We find the asymptotic form in the neighbourhood of λ_0. We use the results of /11/ for the case when the saddle point is close to the pole. We obtain

$$p_* = \frac{\varepsilon^{-1/2}}{2x} \frac{\exp[\varepsilon^{-1/4}\alpha^{-1}t_* f(\theta_5)]}{\sqrt{1+\theta_5 g(\theta_5)}} \exp\left\{[\theta_1(\lambda) - \theta_5]^2 \times \frac{t_* k(\theta_5)\varepsilon^{-1/4}}{2\alpha}\right\}\left\{1 - \operatorname{erf}\left[(\theta_1(\lambda) - \theta_5)\sqrt{\frac{t_* k(\theta_5)\cdot\varepsilon^{-1/4}}{2\alpha}}\right]\right\} \tag{4.4}$$

Let us show that (4.4) gives an asymptotic form for the pressure which is everywhere suitable if $\theta_1 \geqslant \theta_5$, if we replace $\theta_1(\lambda_0) = \theta_5$ by $\theta_1(\lambda)$.

Using the asymptotic form $(1 - \operatorname{erf} z) e^{z^2} \sim z^{-1}\sqrt{\pi}$ when $z \gg 1$, we rewrite (4.4) as

$$p_* = \frac{\varepsilon^{-1/2}}{x\sqrt{\pi}} \frac{\exp[\varepsilon^{-1/4}\alpha^{-1}t_* f(\theta_1(\lambda))]}{\sqrt{1+\theta_1 g(\theta_1)}} \times \int_0^\infty \exp\left\{-\tau\left[\tau + (\theta_1(\lambda) - \theta_5)\sqrt{\frac{t_* k(\lambda)\varepsilon^{-1/4}}{2\alpha}}\right]\right\} d\tau \tag{4.5}$$

When $\theta_1 > \theta_5$, (4.5) becomes (4.2). With $\theta_1 = \theta_5$, the integral in (4.5) is considered and we obtain asymptotic form (4.3). With $\theta_1 < \theta_5$, (4.5) gives the same asymptotic form (4.2) without taking account of the residue. In this case, however, the contribution of the saddle point is small compared with that of the pole. For, it can be shown that the function $\theta_1(\lambda)$ is increasing, while at the saddle point we have

$$\frac{df(\theta_1(\lambda), \lambda)}{d\lambda} = -\frac{\theta_1(\lambda)}{\sqrt{1+\theta_1(\lambda)}}.$$

Hence it is clear that $\theta_1 = 0$ is the maximum point of $f(\theta_1(\lambda), \lambda)$, while with $\theta_1 < \theta_5 < 0$ the function is increasing and $f(\theta_1, \lambda) < f(\theta_5, \lambda_0)$. Hence the contribution of the saddle point is small.

The function $f(\theta_1, \lambda(\theta_1)) < 0$ for all $\theta_1 \neq 0$, $f(0, \lambda(0)) = 0$, $\lambda(0) = 1$. In the intermediate zone, therefore, the asymptotic form of the pressure p^0 is everywhere exponentially small, except for the zone around $\lambda = 1$. This zone corresponds to $\theta_1 > \theta_5$, i.e., to asymptotic form (4.2), whence it is clear that the zone of values of p_* which are not exponentially small is given by the condition $f(\theta_1(\lambda), \lambda) \sim \varepsilon^{1/4}$. In this zone we have the expansion $\theta_1(\lambda) = \lambda - 1 + \ldots$, $f = -(\lambda^2 - 1)/2 + \ldots$. Finally, in the zone about $\lambda = 1$ we obtain the asymptotic form of the pressure (2.2) as

$$p_* = \frac{\varepsilon^{-1/4} t_*^{-1/2}}{\sqrt{2\pi\alpha}} \exp\left\{\frac{-[(x-1)\varepsilon^{1/4} - t_*]^2}{2\alpha t_* \varepsilon^{1/4}}\right\}.$$

Hence it can be seen that the union of pressures (3.9) and (2.2) occurs when $\delta = \varepsilon^{1/4}$, $N = 1/\sqrt{2\pi\alpha}$. It can be shown that the union of velocities (3.9) and (2.2) gives the same result.

The union of (3.9) and (3.6) gives the condition $p_e = u_e$ for $x_e = t$. Hence $F_2 \equiv 0$. The union of velocity (3.6) and the velocity around the piston $u = C/x^2$ gives the relation $F_1(t) = C(t)$.

To find $g(t)$ in (3.3), we unite the pressure (3.3) and the pressure (3.6). The intermediate coordinates are $t \sim 1$, $x = x_*\varepsilon^{-1/4} \sim \varepsilon^{-1/4}$. From (3.4) (for $\varkappa = 0$) we obtain in the principal approximation: $n \sim \varepsilon^{-3/4}$, $G \sim \varepsilon$. The asymptotic form of the pressure (3.3) is

$$p_* = \varepsilon^{-1/4}[g(t) - \sigma], \quad \sigma(1, G) \sim G \sim \varepsilon.$$

The asymptotic form of the pressure (3.6) is $p_* = \varepsilon^{1/4}C'(t)/x_*$. Hence it follows that $g(t) = 0$. To perform the union when $\varkappa \neq 0$, we have to study the properties of the solution of Eq. (3.1).

We write the component expansions, suitable everywhere for $t \sim 1$:

$$u = \frac{C(-x\varepsilon^{1/2} + t)}{x^2} + \frac{C'(-x\varepsilon^{1/2} + t)}{x} + \frac{\varepsilon^{1/4}t^{-1/2}}{\sqrt{2\pi\alpha}} \exp\left[\frac{-(x - 1 - t\varepsilon^{-1/2})^2\varepsilon^{1/2}}{2t\alpha}\right]$$

$$p = \frac{C'(-x\varepsilon^{1/2} + t)}{x} - G + \frac{\varepsilon^{-1/4}t^{-1/2}}{\sqrt{2\pi\alpha}} \exp\left[\frac{-(x - 1 + t\varepsilon^{-1/2})^2\varepsilon^{1/2}}{2t\alpha}\right].$$

The pressure $\sigma = \sigma(1 + \varepsilon p, s)$.

5. Determination of the dynamic characteristics of the flow at a piston.

The solution obtained enables us to find simple time dependences of the pressure and stress at the piston when $t \sim \varepsilon^{1/2}$. Noting that the internal energy in our medium depends weakly on the entropy when $s \sim 1$, we can put $\sigma = p$ in the principal approximation in the initial zone. With $x = 1$ the integrals in (2.2) can be evaluated by means of residues. For the pressure σ, and the radial and angular components of the stress tensor at the piston, we obtain

$$\sigma = \frac{\varepsilon^{-1/2}}{\sqrt{1+\theta_5}\, g(\theta_5)} e^{\tau\theta_5/\alpha}$$

$$\sigma_x = -\varepsilon^{-1/2}\left[4\alpha + \frac{1-2\alpha\sqrt{1+\theta_5}}{g(\theta_5)\sqrt{1+\theta_5}} e^{\tau\theta_5/\alpha}\right]$$

$$\sigma_\varphi = \varepsilon^{-1/2}\left[2\alpha + \frac{1}{g(\theta_5)}\left(2\alpha + \frac{2\alpha^2\sqrt{1+\theta_5}}{\theta_5} - \frac{1}{\sqrt{1+\theta_5}}\right) e^{\tau\theta_5/\alpha}\right]$$

In Fig.2 we plot curves $\varepsilon^{1/2}\sigma(\alpha)$, $\varepsilon^{1/2}\sigma_x(\alpha)$, $\varepsilon^{1/2}\sigma_\varphi(\alpha)$ at the initial instant $\tau = 0$ (curves *1–3* respectively). With $\alpha > 2$, the curves are well described by the asymptotic forms $\varepsilon^{1/2}\sigma_x = -4\alpha$, $\varepsilon^{1/2}\sigma_\varphi = 2\alpha$. As α increases, the initial pressure tends to zero as $2/\alpha$, but the stresses increase rapidly due to the viscous components.

It is interesting that exponential decay is obtained for the pressure at the piston as τ increases for any α. For the stresses, exponential decay occurs only for small α. The dimensionless decay constant $b = -\alpha/\theta_5$, relative to $\varepsilon^{1/2}$, is shown by curve *4* of Fig.2. It was confirmed theoretically in /2/ that the peak approximation for an inviscid fluid in the case of spherical symmetry gives the pressure at the piston falling exponentially with time, the theoretical value of the decay constant being equal to unity. It is easily seen that, in the viscous medium, $b(\alpha) \to 1$ as $\alpha \to 0$.

It can be seen from Fig.2 that the angular component of the stress tensor σ_φ at the initial instant is positive roughly for $\alpha \geqslant 0.5$.

Through passage to the limit as $\alpha \to 0$ is impossible in our solution in the zone of the forming SW, in the short time zone $t \sim \varepsilon^{1/2}$ we can pass to the limit in (2.2) as $\tau \sim 1$, $\alpha \to 0$. This gives

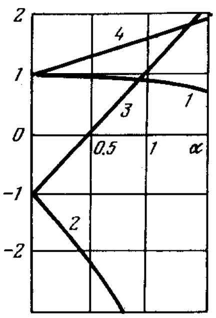

Fig.2

$$u = p\varepsilon^{1/2} = \frac{1}{2x}\left[1 - \operatorname{erf}\left((\lambda - 1)\sqrt{\frac{\tau}{2\alpha}}\right)\right] \tag{5.1}$$

It is clear from this that the viscosity must be taken into account in the principal approximation when $t \sim \varepsilon^{1/2}\alpha$ (with these times, solution (2.2) must be used) and in the SW zones $(\lambda - 1)\sqrt{\tau/2\alpha} \sim 1$. The viscous profile of SW, constructed in the present paper, is important for $x_0 \sim \zeta/(\bar{\rho}_0 U_0)$. For instance, for water with $U_0 \sim 10$ m/sec., this corresponds to $x_0 \sim 10^{-7}$ m. With $x_0 \gg \zeta(\bar{\rho}_0 U_0)$ the profile, valid for an inviscid medium /2/, is obtained. The effect of viscosity makes itself felt here by the jump being smeared in the form of the boundary layer function (5.1).

6. Conclusions.

Our solution shows that, for small Reynolds numbers, $\mathrm{Re} = 1/\alpha^0 \sim \varepsilon^{1/2}$, an SW profile is formed from the impact which is different from the profile in an inviscid fluid. The profile immediately transforms into a Gaussian profile, whose width increases as the square root of the time. The amplitude of the spherical SW falls in inverse proportion to $t^{3/2}$, as distinct from the inviscid SW, where the fall is inversely proprotional to the time. In the SW zone the disturbances are one order larger than in the wave zone.

The solution in the SW zone is described by a parabolic equation, and in the wave zone, by a hyperbolic system. The characteristics of this system are subcharacteristics of system (1.1) for the problem /12/. The boundary layer appears on the front characteristic of the linearized inviscid flow. Notice that our solution describes, not the usual viscous boundary layer resulting as $\mathrm{Re} \to \infty$, but the boundary layer arising as a result of the weak compressibility of the medium when $\mathrm{Re} \ll 1$.

As distinct from the SW profile in an ideal fluid, the geometric similarity of the SW profile in a viscous medium is destroyed as a result of the extra dependence on the Reynolds number.

In the neighbourhood of the contact boundary, as a result of the viscous components of the stress tensor at short times the continuity of the medium can be destroyed, since positive stresses can arise.

At short times, the pressure on the contact boundary falls exponentially with time, as in an inviscid medium, i.e., the peak approximation also holds for a strongly viscous medium in the case of spherical symmetry.

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ON LAMINAR PRESEPARATION FLOW*

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The boundary layer of an incompressible fluid in the domain ahead of the departure of the free streamline from the surface of a smooth body or a break-point of its generator, is considered. The potential of the external irrotational velocity field is taken from the theory of jet flows. It is assumed with respect to the initial value of the surface friction that its order can vary over a wide range, while remaining finite, or taking extremely large values. The boundary layer in the preseparation domain always admits of a unified mathematical treatment, in which the initial surface friction plays the role of a parameter.

1. External potential flow. For measuring both the independent and the required quantities we take a system of units in which the basis quantities are the radius of curvature of the body generator at the point of separation, the velocity of the external potential flow at this point, and the fluid density. Changing to dimensionless variables, we direct the s axis of the curvilinear orthogonal system of coordinates along the body generator, and the n axis along the normal to it. Let u', v' be the components of the disturbing velocity vector, and p' the excess pressure in the external potential flow domain. In accordance with the linearized form of the Bernoulli integral, $u' = -p'$, while the complex velocity is $-(p' + iv')$. By the theory of jet flows of an ideal incompressible fluid, we know that, in the neighbourhood of the departure point of the free streamline from the body /1/

$$p' + iv' = ib_{1/2}z^{1/2} + ib_{3/2}z^{3/2} + \ldots, \quad z = s + in \tag{1.1}$$

When $\arg z \to 0$, the pressure $p' \to 0$, whereas

$$v' = b_{1/2}s^{1/2} + b_{3/2}s^{3/2} + \ldots \tag{1.2}$$

If $\arg z \to \pi$, then $v' \to 0$, while

$$p' \to -b_{1/2}(-s)^{1/2} + b_{3/2}(-s)^{3/2} + \ldots \tag{1.3}$$

In accordance with (1.2), the equation of the free streamline is

$$n = \frac{2}{3}b_{1/2}s^{3/2} + \frac{2}{5}b_{3/2}s^{5/2} + \ldots \tag{1.4}$$

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